Are You Positive? Solutions

Part 1

Solution: *The expression is positive for values of x that are less than -6 or between 2.5 and 7*

Method 1: Compound Inequalities

I first realized that it didn't really matter which binomial I put where. Basically I have three binomials that are being multiplied and divided together, so for the overall expression to be positive, either all three individual binomials must be positive or two of them must be negative and the third one be positive. Thinking of positive as meaning greater than 0, I wrote inequalities for each binomial to represent each of those cases:

If all three are positive, then:

$$\begin{aligned} 5-2x &> 0 \\ 5 &> 2x \\ \frac{5}{2} &> x \\ x &< 2.5 \end{aligned} \quad \text{and} \quad \begin{aligned} x-7 &> 0 \\ x &> 7 \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{x}{2}+3 &> 0 \\ \frac{x}{2} &> -3 \\ x &> -6 \end{aligned}$$

Since all three inequalities must be true, I need x to be larger than 7 and at the same time smaller than 2.5, which is clearly impossible. So, there are no x values that will make all three binomials be positive.

To make one of the binomials be negative, I just needed to switch the original inequality so that it would be less than 0 rather than greater than 0. So, for the first one to be positive and the second two to be negative, I'd have:

$$\begin{aligned} 5-2x &> 0 \\ 5 &> 2x \\ \frac{5}{2} &> x \\ x &< 2.5 \end{aligned} \quad \text{and} \quad \begin{aligned} x-7 &< 0 \\ x &< 7 \end{aligned} \quad \text{and} \quad \begin{aligned} \frac{x}{2}+3 &> 0 \\ \frac{x}{2} &> -3 \\ x &> -6 \end{aligned}$$

Again I need all three to be true, so x must be smaller than all three of the final numbers. The only way that happens is if it's smaller than the smallest of the three, so x must be less than -6. For any x less than -6, the 5 – 2x will be positive and each of the others will be negative, making the overall value positive.

I noticed that my solving steps were the same and the only difference was how the final sign came out. So to try the next two cases, I just flipped the signs of two of the original results to make them negative. That saved some work!

So, for the second one to be positive and the first and third to be negative, I'd have:

$$x > 2.5 \quad \text{and} \quad x > 7 \quad \text{and} \quad x < -6$$

Again, there is no way x can be greater than 7 and at the same time less than -6, so there is no x value that makes the second binomial positive and the first and third negative.

Finally, to have the third one positive and the first two negative, I'd have:

$$x > 2.5 \quad \text{and} \quad x < 7 \quad \text{and} \quad x > -6$$

Since x has to be greater than both -6 and 2.5, that means it has to be greater than 2.5. But it also has to be less than 7, so I wind up with the interval between 2.5 and 7 as the values that make the third binomial be positive and the first two be negative, making the overall value positive.

The values of x that make the expression positive are between 2.5 and 7 or less than -6.

Method 2: Graphing the Individual Binomials

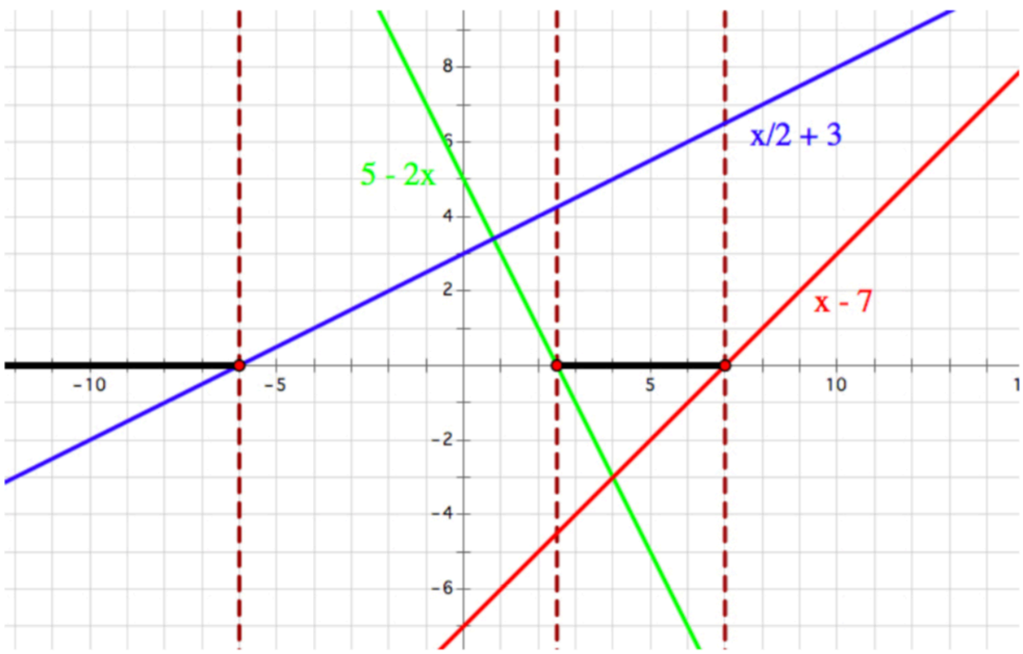
I noticed that all three of the binomials were linear expressions of the form mx + b. I decided to graph each of them on the same set of axes because then I could see where each one was positive (above the x-axis) and negative (below the x-axis). I started by finding the slope and y-intercept for each one:

$5-2x$ has a y-intercept of 5 and a slope of -2

x-7 has a y-intercept of -7 and a slope of 1

$\frac{x}{2}+3$ has a y-intercept of 3 and a slope of $\frac{1}{2}$

I plotted all three lines on the same set of axes. I also drew vertical lines at the spots where the lines hit the x-axis, which were at -6, 2.5, and 7. Looking at the graph, I was able to solve the problem:



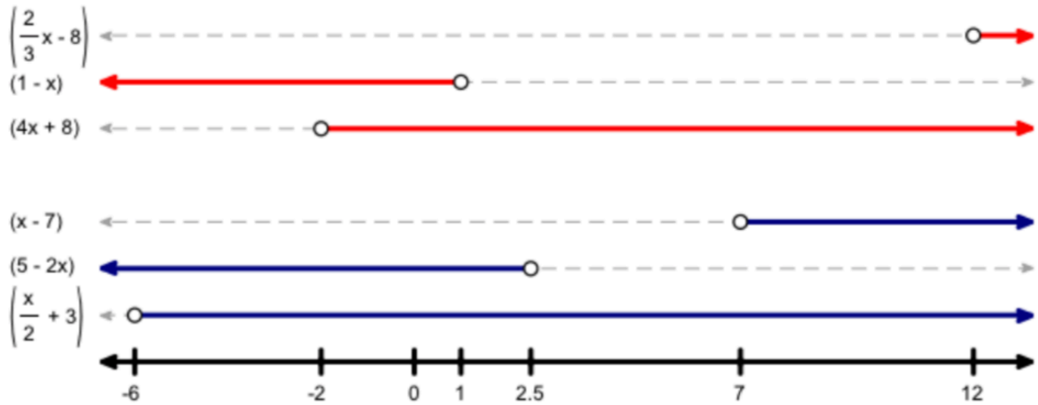
As I said, if the graph is above the x-axis, it's positive and if it's below the axis, it's negative. So $\frac{x}{2}+3$, for instance, is negative to the left of -6 and positive to the right of it. Similarly, 5 – 2x is positive to the left of 2.5 and negative to the right of it. Finally, x – 7 is negative to the left of 7 and positive to the right of it.

The vertical lines split the x-axis into four intervals; left of -6, between -6 and 2.5, between 2.5 and 7, and to the right of 7. I can see that to the left of -6, $\frac{x}{2}+3$ and x – 7 are both negative while 5 – 2x is positive. Since the expression consists of those three binomials being multiplied and divided, the two negatives and one positive make an overall positive. I noted that with a heavy black shading on the x-axis in that interval.

Between -6 and 2.5, both $\frac{x}{2}+3$ and 5 – 2x are positive while x – 7 is negative. Those two positives and one negative make the overall expression negative. Between 2.5 and 7, both 5 – 2x and x – 7 are negative while $\frac{x}{2}+3$ is positive. Those two negatives and one positive make the overall expression positive, so again I noted that interval on the x-axis. Finally, to the right of 7 both $\frac{x}{2}+3$ and x – 7 are positive while 5 – 2x is negative. The two positives and one negative make the overall sign negative.

The values of x that make the expression positive are between 2.5 and 7 or less than -6.

Part 2
Solution: Trade in $x - 7$ for $4x + 8$ and that would result in an interval of 2.5 units of positive x values that make the overall expression positive.



Now I looked at the chart and tried in my mind to see what would happen swapping various binomials. Since I'm looking for the smallest interval of positive x values, I really only had to pay attention to what was to the right of 0 on my number line. I looked for ways to make three positives or one positive and two negatives. I also knew that the interval from the original problem was from 2.5 to 7, or 4.5 units, so I need to find something shorter than that.

I realized I can't use a combination where I have three positives or two negatives and a positive on the right end, or there would be an infinite number of positive x values that work. That made it a lot easier to eliminate some of the choices. For example, 1 – x could only be traded with 5 – 2x. I tried that first and found that between x = 1 and x = 7 there would be two negatives and a positive, so that interval is 6 units and too big.

I continued looking at options and found that if I traded x – 7 for 4x + 8 I would have three positives between 0 and 2.5, and two positives and a negative for anything to the right of 2.5. That gives an interval from 0 to 2.5, which is 2.5 units. I think that's the smallest I can find.